

Quantum Nonlocality Enhanced by Homogenization

Xu Chen,¹ Hong-Yi Su,^{1,*} Zhen-Peng Xu,¹ Yu-Chun Wu,² and Jing-Ling Chen^{1,3,†}

¹*Theoretical Physics Division, Chern Institute of Mathematics,
Nankai University, Tianjin 300071, People's Republic of China*

²*Key Laboratory of Quantum Information, University of Science
and Technology of China, 230026 Hefei, People's Republic of China*

³*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543*

(Dated: January 9, 2015)

Homogenization proposed in [Y.-C Wu and M. Żukowski, Phys. Rev. A 85, 022119 (2012)] is a procedure to transform a tight Bell inequality with partial correlations into a full-correlation form that is also tight. In this paper, we check the homogenizations of two families of n -partite Bell inequalities: the Hardy inequality and the tight Bell inequality without quantum violation. For Hardy's inequalities, their homogenizations bear stronger quantum violation for the maximally entangled state; the tight Bell inequalities without quantum violation give the boundary of quantum and supra-quantum, but their homogenizations do not have the similar properties. We find their homogenization are violated by the maximally entangled state. Numerically computation shows the the domains of quantum violation of homogenized Hardy's inequalities for the generalized GHZ states are smaller than those of Hardy's inequalities.

PACS numbers: 03.65.Ud, 03.30.+p, 03.67.-a

I. INTRODUCTION

The problem of the possibility of a local realistic interpretation of quantum mechanics was first addressed in the discussion between Einstein, Podolsky, Rosen (EPR) [1] and Bohr [2]. In order to settle down the philosophical debate, Bell proposed an experimental scheme in 1964 [3]. Bell's seminal paper contains an inequality, which holds for local realistic correlations but can be violated by quantum mechanical correlations. In the work of Clauser, Horne, Shimony, and Holt (CHSH) [4] a new inequality was derived, which, comparing with the original Bell's expression, can be more applicable to real experimental setups. An inequality with lower-order (partial) correlations is usually called a Clauser-Horne-type (CH-type) Bell inequality, whereas an inequality involving only highest-order (full) correlations is usually referred to as a CHSH-type Bell inequalities.

To identify quantum nonlocality (QN), one needs a set of complementary observables for each party. It is anticipated that the ability to detect QN becomes stronger as the number of observables (settings) increases. For instance, the 2-setting CHSH inequality determines the visibility for the Werner state as $1/\sqrt{2}$, while a 465-setting inequality [5] decreases the visibility to 0.7056, which is slightly smaller than $1/\sqrt{2}$.

On the other hand, although the CHSH inequality can detect the QN for all two-qubit pure entangled states, for more parties it is not desirable to invoke the full-correlation inequality, like the CHSH inequality. In fact, the first Bell inequality to identify QN for the whole

domain of the generalized Greenberger-Horne-Zeilinger (GHZ) state $\cos\theta|000\rangle + \sin\theta|111\rangle$ is comprised of partial correlations.

In general, in constructing Bell inequalities there is a trade-off between full correlations and the ability to detect QN for more generalized states. Wu and Żukowski [6] show how to transform a CH-type inequality into a CHSH-type inequality, the tightness being preserved. In this paper, we use the procedure of homogenization to compare the quantum violation for various states. Through the homogenization, the setting for each party is increased by one. This strengthens the quantum violation for the maximally entangled state, while the violation regimes for the nonmaximally entangled state could be narrowed.

The organization of this paper is as follows. In Sec. II, we briefly review the procedure of homogenization. Then, in Sec. III, we focus on the generalized GHZ state and compare its quantum violation of the Hardy inequality before and after the homogenization. Likewise, in Sec. IV we study the influence of the homogenization on a family of tight Bell inequalities without quantum violation. We discuss the results in Sec. V and propose a possible reason for the enhanced QN by the homogenization.

II. BRIEF REVIEW ON HOMOGENIZATION

In general, the CHSH- and CH-type inequalities read

$$\langle \sum_{ij} \omega_{ij} a_i b_j \rangle \leq 1, \quad (1)$$

*Electronic address: hysu@mail.nankai.edu.cn

†Electronic address: chenjl@nankai.edu.cn

$$0 \leq I(\hat{a}, \hat{b}) = \langle c + \sum_i \alpha_i a_i + \sum_j \beta_j b_j + \sum_{ij} \gamma_{ij} a_i b_j \rangle \leq M, (2)$$

where ω_{ij} , α_i , β_j and γ_{ij} are real coefficients, c is a real constant, M is the classical upperbound of (2), and a_i, b_j are dichotomic observables taking values ± 1 . Wu and Żukowski [6] showed that (2) can be transformed into (1) by homogenization, and that if (2) is tight then (1) is also tight. Specifically, the homogeneous expression is obtained as

$$H(I) = \langle c' a_0 b_0 + \sum_i \alpha_i a_i b_0 + \sum_j \beta_j b_j a_0 + \sum_{ij} \gamma_{ij} a_i b_j \rangle, (3)$$

with

$$-\frac{M}{2} \leq \frac{1}{a_0 b_0} H(I) \leq \frac{M}{2}, \quad c' = c - M/2. \quad (4)$$

III. HARDY'S NONLOCALITY INEQUALITIES

Cereceda [7] extend Hardy's nonlocality proof for two spin-1/2 particles [8] to the case of n spin-1/2 particles configured in the generalized GHZ state. We now show that the maximal quantum violation of the homogenized Hardy's inequality is stronger than Hardy's original inequality. The n -qubit CH-type Hardy's inequality reads [7]

$$\begin{aligned} & p(0_1 0_2 0_3 \cdots 0_n | 0_1 0_2 0_3 \cdots 0_n) \\ & \leq p(1_1 1_2 1_3 \cdots 1_n | 1_1 1_2 1_3 \cdots 1_n) \\ & + p(0_1 0_2 0_3 \cdots 0_n | 1_1 0_2 0_3 \cdots 0_n) \\ & + p(0_1 0_2 0_3 \cdots 0_n | 0_1 1_2 0_3 \cdots 0_n) \\ & + \cdots + p(0_1 0_2 0_3 \cdots 0_n | 1_1 0_2 0_3 \cdots 1_n), \end{aligned} \quad (5)$$

where $p(i_1 i_2 i_3 \cdots i_n | j_1 j_2 j_3 \cdots j_n)$ denotes the joint probability of obtaining result i_k under setting j_k for the k -th qubit, with $i_k, j_k = 0, 1$ and k running from 1 to n . In the following context the subscript for each party could be omitted with no confusion. Let us rewrite (5) in the form of correlations. For $n = 3$,

$$\begin{aligned} & |5 - A_1 - B_1 - C_1 - A_2 B_1 - A_2 C_1 - A_1 B_2 - B_2 C_1 \\ & - A_1 C_2 - B_1 C_2 - A_2 B_2 - A_2 C_2 - B_2 C_2 + A_1 B_1 C_1 \\ & + A_2 B_2 C_2 - A_2 B_1 C_1 - A_1 B_2 C_1 - A_1 B_1 C_2|/8 \leq 1, \end{aligned} \quad (6)$$

where $A_1 = p(0_1 | 0_1) - p(1_1 | 0_1)$ and $A_2 = p(0_1 | 1_1) - p(1_1 | 1_1)$ are observables for the first party, and likewise for B_i, C_j . It is violated by the three-qubit GHZ state by a factor of 1.1755, and it is violated maximally by the state $0.8393|000\rangle + 0.5435|100\rangle + 0.01283|101\rangle - 0.0009074|110\rangle + 0.002652|111\rangle$ by a factor of 1.236. In presence of noise, the state is a mixed state defined by $\rho = V|GHZ\rangle\langle GHZ| + (1 - V)\rho_{\text{noise}}$, where V is called visibility, $\rho_{\text{noise}} = \frac{1}{8}\mathbb{I}^{\otimes 3}$ for three qubits, and \mathbb{I} is the 2×2

identity matrix. For the GHZ state, the threshold visibility V is 0.6812. The homogenized inequality can be written as

$$\begin{aligned} & |5A_0 B_0 C_0 - A_1 B_0 C_0 - A_0 B_1 C_0 - A_0 B_0 C_1 - A_2 B_1 C_0 \\ & - A_2 B_0 C_1 - A_1 B_2 C_0 - A_0 B_2 C_1 - A_1 B_0 C_2 - A_0 B_1 C_2 \\ & - A_2 B_2 C_0 - A_2 B_0 C_2 - A_0 B_2 C_2 + A_1 B_1 C_1 + A_2 B_2 C_2 \\ & - A_2 B_1 C_1 - A_1 B_2 C_1 - A_1 B_1 C_2|/8 \leq 1, \end{aligned} \quad (7)$$

It is violated maximally by the three-qubit GHZ state by a factor of 1.8 (see Fig. 1), the threshold visibility V is 0.5556.

For $n = 4$,

$$\begin{aligned} & |20 - 2(A_1 + B_1 + C_1 + D_1) - A_1 B_1 - A_1 C_1 - A_1 D_1 \\ & - B_1 C_1 - B_1 D_1 - C_1 D_1 - A_2 B_1 - A_2 C_1 \\ & - A_2 D_1 - A_1 B_2 - B_2 C_1 - B_2 D_1 - A_1 C_2 \\ & - B_1 C_2 - C_2 D_1 - A_1 D_2 - B_1 D_2 - C_1 D_2 \\ & - A_2 B_2 - A_2 C_2 - A_2 D_2 - B_2 C_2 - B_2 D_2 - C_2 D_2 \\ & - A_2 B_1 C_1 - A_2 B_1 D_1 - A_2 C_1 D_1 - A_1 B_2 C_1 \\ & - A_1 B_2 D_1 - B_2 C_1 D_1 - A_1 B_1 C_2 - A_1 C_2 D_1 \\ & - B_1 C_2 D_1 - A_1 B_1 D_2 - A_1 C_1 D_2 - B_1 C_1 D_2 \\ & + A_2 B_2 C_2 + A_2 B_2 D_2 + A_2 C_2 D_2 + B_2 C_2 D_2 \\ & + A_1 B_1 C_1 D_1 - A_1 B_2 C_1 D_1 - A_2 B_2 C_2 D_2 \\ & - A_1 B_1 C_2 D_1 - A_1 B_1 C_1 D_2 - A_2 B_1 C_1 D_1|/24 \leq 1, \end{aligned} \quad (8)$$

It is violated by the four-qubit GHZ state by a factor of 1.0690, it is violated maximally by a certain state by a factor of 1.1665. On account of noise, we consider a mixed state similar to the three-qubit case as defined below (6). Here $\rho_{\text{noise}} = \frac{1}{16}\mathbb{I}^{\otimes 4}$ for four qubits, the threshold visibility V is 0.7071 for the GHZ state. The homogenized inequality can be written as

$$\begin{aligned} & |20A_0 B_0 C_0 D_0 - 2(A_1 B_0 C_0 D_0 + A_0 B_1 C_0 D_0 \\ & + A_0 B_0 C_1 D_0 + A_0 B_0 C_0 D_1) - A_1 B_1 C_0 D_0 \\ & - A_1 B_0 C_1 D_0 - A_1 B_0 C_0 D_1 - A_0 B_1 C_1 D_0 \\ & - A_0 B_1 C_0 D_1 - A_0 B_0 C_1 D_1 - A_2 B_1 C_0 D_0 \\ & - A_2 B_0 C_1 D_0 - A_2 B_0 C_0 D_1 - A_1 B_2 C_0 D_0 \\ & - A_0 B_2 C_1 D_0 - A_0 B_2 C_0 D_1 - A_1 B_0 C_2 D_0 \\ & - A_0 B_1 C_2 D_0 - A_0 B_0 C_2 D_1 - A_1 B_0 C_0 D_2 \\ & - A_0 B_1 C_0 D_2 - A_0 B_0 C_1 D_2 - A_2 B_2 C_0 D_0 \\ & - A_2 B_0 C_2 D_0 - A_2 B_0 C_0 D_2 - A_0 B_2 C_2 D_0 \\ & - A_0 B_2 C_0 D_2 - A_0 B_0 C_2 D_2 - A_2 B_1 C_1 D_0 \\ & - A_2 B_1 C_0 D_1 - A_2 B_0 C_1 D_1 - A_1 B_2 C_1 D_0 \\ & - A_1 B_2 C_0 D_1 - A_0 B_2 C_1 D_1 - A_1 B_1 C_2 D_0 \\ & - A_1 B_0 C_2 D_1 - A_0 B_1 C_2 D_1 - A_1 B_1 C_0 D_2 \\ & - A_1 B_0 C_1 D_2 - A_0 B_1 C_1 D_2 + A_2 B_2 C_2 D_0 \\ & + A_2 B_2 C_0 D_2 + A_2 B_0 C_2 D_2 + A_0 B_2 C_2 D_2 \\ & + A_1 B_1 C_1 D_1 - A_1 B_2 C_1 D_1 - A_2 B_2 C_2 D_2 \\ & - A_1 B_1 C_2 D_1 - A_1 B_1 C_1 D_2 - A_2 B_1 C_1 D_1|/24 \leq 1, \end{aligned} \quad (9)$$

It is violated maximally by the four-qubit GHZ state by a factor of 1.9524 (see Fig. 1), the threshold visibility V is 0.5121.

IV. BELL INEQUALITIES WITH NO QUANTUM VIOLATION

The quantum correlations (QC) are in general more stronger than classical correlations (CC). Augusiak *et al.* [9] showed how unextendable product bases (UPBs) that satisfy a given requirement give rise to a family of tight Bell inequalities without quantum violation. In these situations, QC and CC perform equally well in information tasks, while the supraquantum nonsignaling correlations do provide an advantage over CC. Thus such inequalities pinpoint the facet of polytope that separate quantum and supraquantum correlations, and provide better understandings of different sets of correlations which serve as valuable information resources.

Bell inequality with non-negative weights q_j , which always can be assumed to obey $0 \leq q_j \leq 1$.

$$\sum_j q_j p(\mathbf{a}_j | \mathbf{x}_j) \leq \max \{q_j\}, \quad (10)$$

From the initial set of orthogonal product vectors, Augusiak *et al.* proved that all these inequalities are not violated by QC [9] [10]. By local unitaries and permutations of particles, all UPBs can be brought to $S_{(i)}^0 \equiv S_0 = \{|0\rangle, |1\rangle\}$ and $S_{(i)}^1 \equiv S_1 = \{|e\rangle, |\bar{e}\rangle\}$ ($i = 1, 2, 3$). We assign conditional probabilities in the following way: $|000\rangle \rightarrow p(000|000)$, $|\bar{1}\bar{e}\bar{e}\rangle \rightarrow p(110|011)$, $|e1\bar{e}\rangle \rightarrow p(011|101)$, $|\bar{e}e1\rangle \rightarrow p(101|110)$. Then, we can get the tight Bell inequality with no quantum violation found in [11] and [12]. For odd n , the inequality can be written as

$$\sum_{k=0}^{(n-1)/2} \sum_{i_1 < \dots < i_{2k}=1}^n T_{i_1 \dots i_{2k}} p(\mathbf{0} | \mathbf{0}) \leq 1, \quad (11)$$

and for even n

$$\sum_{k=0}^{(n-2)/2} \sum_{i_1 < \dots < i_{2k}=2}^n T_{i_1 \dots i_{2k}} [p(\mathbf{0} | \mathbf{0}) + p(0 \dots 01 | 10 \dots 0)] \leq 1. \quad (12)$$

Here $\mathbf{0} = (0, \dots, 0)$, and $T_{i_1 \dots i_{2k}}$ denotes a flip ($0 \leftrightarrow 1$) of input bits and output bits at positions i_1, \dots, i_{2k} and $i_1 - 1, \dots, i_{2k} - 1$ (if $i_j = 1$, then $i_j - 1 = n$), respectively.

For $n = 3$, we obtain the inequality

$$p(000|000) + p(101|110) + p(011|101) + p(110|011) \leq 1. \quad (13)$$

For $n = 4$,

$$\begin{aligned} & p(0000|0000) + p(0001|1000) + p(0110|0011) \\ & + p(0111|1011) + p(1010|0101) + p(1011|1101) \\ & + p(1100|0110) + p(1101|1110) \leq 1. \end{aligned} \quad (14)$$

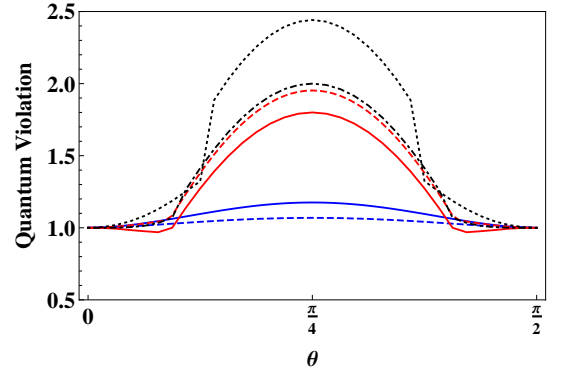


FIG. 1: The quantum violation for the generalized GHZ state. The blue solid, red solid, blue dash, red dash, black dot-dash, and black dot curves correspond to inequalities (6), (7), (8), (9), (16), and (18), respectively.

Rewriting (13), the symbols in inequalities of probability $0 \rightarrow 1$ and $1 \rightarrow 2$ into the correlation function, we obtain

$$\begin{aligned} & |A_1 B_1 - A_2 B_1 + A_1 B_2 - A_2 B_2 + A_1 C_1 + A_2 C_1 \\ & + B_1 C_1 - B_2 C_1 - B_2 C_2 - A_1 C_2 - A_2 C_2 + B_1 C_2 \\ & + A_1 B_1 C_1 + A_2 B_2 C_1 + A_2 B_1 C_2 + A_1 B_2 C_2|/4 \leq 1. \end{aligned} \quad (15)$$

This inequality cannot be violated in quantum mechanics. However, let us see its homogenized inequality:

$$\begin{aligned} & |A_1 B_1 C_0 - A_2 B_1 C_0 + A_1 B_2 C_0 - A_2 B_2 C_0 \\ & + A_1 B_0 C_1 + A_2 B_0 C_1 + A_0 B_1 C_1 - A_0 B_2 C_1 \\ & - A_0 B_2 C_2 - A_1 B_0 C_2 - A_2 B_0 C_2 + A_0 B_1 C_2 \\ & + A_1 B_1 C_1 + A_2 B_2 C_1 + A_2 B_1 C_2 + A_1 B_2 C_2|/4 \leq 1, \end{aligned} \quad (16)$$

which is equivalent to the inequality in Ref. [13]. It is violated maximally by the three-qubit GHZ state by a factor of 2 (see Fig. 1).

For $n = 4$

$$\begin{aligned} & |2A_1 C_1 + 2B_1 C_1 - 2A_1 C_2 + 2A_2 C_1 - 2A_2 C_2 \\ & + 2B_1 C_2 - 2B_2 C_1 - 2B_2 C_2 + A_1 B_1 C_1 + A_2 B_1 C_1 \\ & + A_1 B_2 C_1 + A_2 B_2 C_1 + A_1 B_1 C_2 + A_2 B_1 C_2 + A_1 B_2 C_2 \\ & + A_2 B_2 C_2 + A_1 B_1 D_1 - A_2 B_1 D_1 + A_1 B_2 D_1 - A_2 B_2 D_1 \\ & + A_1 C_1 D_1 - A_2 C_1 D_1 - A_1 C_2 D_1 + A_2 C_2 D_1 - A_1 B_1 D_2 \\ & + A_2 B_1 D_2 - A_1 B_2 D_2 + A_2 B_2 D_2 + A_1 C_1 D_2 - A_2 C_1 D_2 \\ & - A_1 C_2 D_2 + A_2 C_2 D_2 + A_1 B_1 C_1 D_1 - A_2 B_1 C_1 D_1 \\ & + A_1 B_2 C_2 D_1 - A_2 B_2 C_2 D_1 + A_1 B_2 C_1 D_2 - A_2 B_2 C_1 D_2 \\ & + A_1 B_1 C_2 D_2 - A_2 B_1 C_2 D_2|/8 \leq 1, \end{aligned} \quad (17)$$

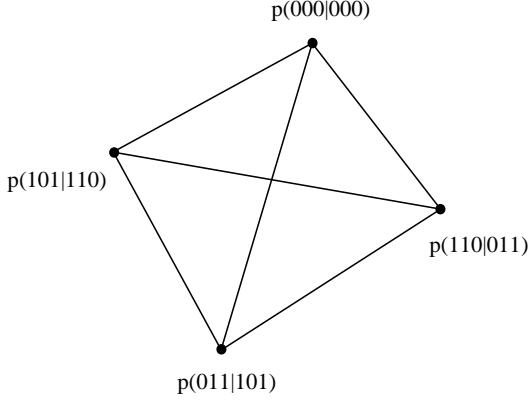


FIG. 2: The exclusivity graph for (13). The sum of probabilities of pairwise exclusive events cannot exceed 1.

whose homogenized inequality can be written as

$$\begin{aligned}
& |2A_1B_0C_1D_0 + 2A_2B_0C_1D_0 + 2A_0B_1C_1D_0 \\
& - 2A_0B_2C_1D_0 - 2A_1B_0C_2D_0 - 2A_2B_0C_2D_0 \\
& + 2A_0B_1C_2D_0 - 2A_0B_2C_2D_0 + A_1B_1C_1D_0 \\
& + A_2B_1C_1D_0 + A_1B_2C_1D_0 + A_2B_2C_1D_0 \\
& + A_1B_1C_2D_0 + A_2B_1C_2D_0 + A_1B_2C_2D_0 \\
& + A_2B_2C_2D_0 + A_1B_1C_0D_1 - A_2B_1C_0D_1 \\
& + A_1B_2C_0D_1 - A_2B_2C_0D_1 + A_1B_0C_1D_1 \\
& - A_2B_0C_1D_1 - A_1B_0C_2D_1 + A_2B_0C_2D_1 \\
& - A_1B_1C_0D_2 + A_2B_1C_0D_2 - A_1B_2C_0D_2 \\
& + A_2B_2C_0D_2 + A_1B_0C_1D_2 - A_2B_0C_1D_2 \\
& - A_1B_0C_2D_2 + A_2B_0C_2D_2 + A_1B_1C_1D_1 \\
& - A_2B_1C_1D_1 + A_1B_2C_2D_1 - A_2B_2C_2D_1 \\
& + A_1B_2C_1D_2 - A_2B_2C_1D_2 + A_1B_1C_2D_2 \\
& - A_2B_1C_2D_2|/8 \leq 1.
\end{aligned} \tag{18}$$

It is violated maximally by the four-qubit GHZ state by

a factor of 2.4413 (see Fig. 1).

V. DISCUSSION AND CONCLUSIONS

To summarize, we have studied the influence of homogenization on enhancing the quantum nonlocality of two families of Bell inequalities. Through the homogenization, the quantum violation of the Hardy inequality has become stronger for the GHZ state, while the parameter domain for violation regime of the generalized GHZ state has been narrowed. On the other hand, quantum violation has appeared again for the family of tight Bell inequalities without quantum violation. The reason for this is that a tight Bell inequality without quantum violation can be represented by a *complete* graph [14] (see Fig. 2), i.e., event probabilities are pairwise exclusive, so that, according to graph theory, the independence number is equal to the Lovász number [15], indicating the coincidence of the classical upperbound with the quantum maximum. However, by homogenization, it is transformed into a full-correlation form, whose corresponding graph is no longer complete. In this regard, the independence number is in general less than the Lovász number, rendering a quantum violation possible.

Acknowledgments

This work is supported by the National Basic Research Program (973 Program) of China (Grant Nos. 2012CB921900, 2011CBA00200, and 2011CB921200) and the NSF of China (Grant Nos. 11175089, 11475089, 10974193, 11275182, and 60921091). J.L.C. is partly supported by the National Research Foundation and the Ministry of Education, Singapore.

-
- [1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
 - [2] N. Bohr, Phys. Rev. **48**, 696 (1935).
 - [3] J. S. Bell, Physics (Long Island City, N.Y.) **1**, 195 (1964).
 - [4] J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett. **23**, 880 (1969).
 - [5] T. Vértesi, Phys. Rev. A **78**, 032112 (2008).
 - [6] Y.-C. Wu and M. Żukowski, Phys. Rev. A **85**, 022119 (2012).
 - [7] J. L. Cereceda, Phys. Lett. A **327**, 433 (2004).
 - [8] L. Hardy, Phys. Rev. Lett. **71**, 1665 (1993).
 - [9] R. Augusiak, J. Stasinska, C. Hadley, J.K. Korbicz, M. Lewenstein, and A. Acín, Phys. Rev. Lett. **107**, 070401 (2011).
 - [10] R. Augusiak *et al.*, Phys. Rev. A **85**, 042113 (2012).
 - [11] C. Śliwa, Phys. Lett. A **317**, 165-168 (2003).
 - [12] M. L. Almeida, J. D. Bancal, N. Brunner, A. Acín, N. Gisin, and S. Pironio, Phys. Rev. Lett. **104**, 230404 (2010).
 - [13] M. Wieśniak, P. Badziag, and M. Żukowski, Phys. Rev. A **76**, 012110 (2007).
 - [14] A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. **112**, 040401 (2014).
 - [15] L. Lovász, IEEE Trans. Inf. Theory **25**, 1 (1979).